28.6 Relativistic Energy

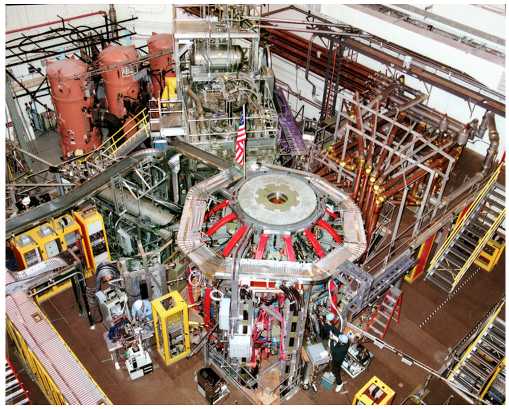

Figure 28.20 The National Spherical Torus Experiment (NSTX) has a fusion reactor in which hydrogen isotopes undergo fusion to produce helium. In this process, a relatively small mass of fuel is converted into a large amount of energy. (credit: Princeton Plasma Physics Laboratory)

A tokamak is a form of experimental fusion reactor, which can change mass to energy. Accomplishing this requires an understanding of relativistic energy. Nuclear reactors are proof of the conservation of relativistic energy.

Conservation of energy is one of the most important laws in physics. Not only does energy have many important forms, but each form can be converted to any other. We know that classically the total amount of energy in a system remains constant. Relativistically, energy is still conserved, provided its definition is altered to include the possibility of mass changing to energy, as in the reactions that occur within a nuclear reactor. Relativistic energy is intentionally defined so that it will be conserved in all inertial frames, just as is the case for relativistic momentum. As a consequence, we learn that several fundamental quantities are related in ways not known in classical physics. All of these relationships are verified by experiment and have fundamental consequences. The altered definition of energy contains some of the most fundamental and spectacular new insights into nature found in recent history.

Total Energy and Rest Energy

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Einstein showed that the law of conservation of energy is valid relativistically, if we define energy to include a relativistic factor.

Total Energy

Total energy E is defined to be

$$E = \gamma mc^2, \tag{28.43}$$

where m is mass, c is the speed of light, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, and v is the velocity of the mass relative to an observer. There are many aspects of the total energy E that we will discuss—among them are how kinetic and potential energies are included in E, and how E is related to relativistic momentum. But first, note that at rest, total energy is not zero. Rather, when $v = 0$, we have $\gamma = 1$, and an object has rest energy.

Rest Energy

Rest energy is

$$E_0 = mc^2. \tag{28.44}$$

This is the correct form of Einstein's most famous equation, which for the first time showed that energy is related to the mass of an object at rest. For example, if energy is stored in the object, its rest mass increases. This also implies that mass can be destroyed to release energy. The implications of these first two equations regarding relativistic energy are so broad that they were not completely recognized for some years after Einstein published them in 1907, nor was the experimental proof that they are correct widely recognized at first. Einstein, it should be noted, did understand and describe the meanings and implications of his theory.

EXAMPLE 28.6

Calculating Rest Energy: Rest Energy is Very Large

Calculate the rest energy of a 1.00-g mass.

Strategy

One gram is a small mass—less than half the mass of a penny. We can multiply this mass, in SI units, by the speed of light squared to find the equivalent rest energy.

Solution

1. Identify the knowns. $m = 1.00 \times 10^{-3}\ \text{kg}; c = 3.00 \times 10^{8}\ \text{m/s}$
2. Identify the unknown. E_0
3. Choose the appropriate equation. $E_0 = mc^2$
4. Plug the knowns into the equation.

$$\begin{aligned} E_0 &= mc^2 = (1.00 \times 10^{-3}\ \text{kg})(3.00 \times 10^{8}\ \text{m/s})^2 \\ &= 9.00 \times 10^{13}\ \text{kg} \cdot \text{m}^2/\text{s}^2 \end{aligned} \tag{28.45}$$

5. Convert units.
   Noting that $1\ \text{kg} \cdot \text{m}^2/\text{s}^2 = 1\ \text{J}$, we see the rest mass energy is

$$E_0 = 9.00 \times 10^{13}\ \text{J}. \tag{28.46}$$

Discussion

This is an enormous amount of energy for a 1.00-g mass. We do not notice this energy, because it is generally not available. Rest energy is large because the speed of light c is a large number and c^2 is a very large number, so that mc^2 is huge for any macroscopic mass. The $9.00 \times 10^{13}\ \text{J}$ rest mass energy for 1.00 g is about twice the energy released by the Hiroshima atomic bomb and about 10,000 times the kinetic energy of a large aircraft carrier. If a way can be found to convert rest mass energy into some other form (and all forms of energy can be converted into one another), then huge amounts of energy can be obtained from the destruction of mass.

Today, the practical applications of *the conversion of mass into another form of energy*, such as in nuclear weapons and nuclear power plants, are well known. But examples also existed when Einstein first proposed the correct form of relativistic energy, and he did describe some of them. Nuclear radiation had been discovered in the previous decade, and it had been a mystery as to where its energy originated. The explanation was that, in certain nuclear processes, a small amount of mass is destroyed and energy is released and carried by nuclear radiation. But the amount of mass destroyed is so small that it is difficult to detect that any is missing. Although Einstein proposed this as the source of energy in the radioactive salts then being studied, it was many years before there was broad recognition that mass could be and, in fact, commonly is converted to energy. (See Figure 28.21.)

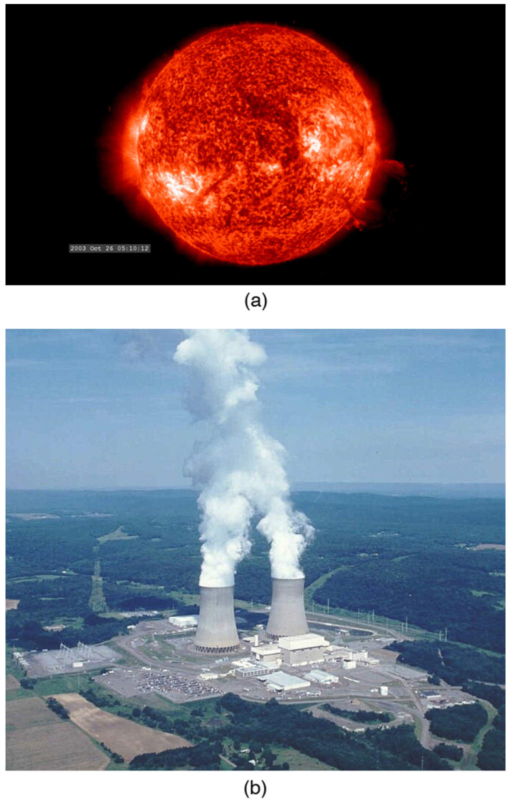

Figure 28.21 The Sun (a) and the Susquehanna Steam Electric Station (b) both convert mass into energy—the Sun via nuclear fusion, the electric station via nuclear fission. (credits: (a) NASA/Goddard Space Flight Center, Scientific Visualization Studio; (b) U.S. government)

Because of the relationship of rest energy to mass, we now consider mass to be a form of energy rather than something separate. There had not even been a hint of this prior to Einstein's work. Such conversion is now known to be the source of the Sun's energy, the energy of nuclear decay, and even the source of energy keeping Earth's interior hot.

Stored Energy and Potential Energy

What happens to energy stored in an object at rest, such as the energy put into a battery by charging it, or the energy stored in a toy gun's compressed spring? The energy input becomes part of the total energy of the object and, thus, increases its rest mass. All stored and potential energy becomes mass in a system. Why is it we don't ordinarily notice this? In fact, conservation of mass (meaning total mass is constant) was one of the great laws verified by 19th-century science. Why was it not noticed to be incorrect? The following example helps answer these questions.

EXAMPLE 28.7

Calculating Rest Mass: A Small Mass Increase due to Energy Input

A car battery is rated to be able to move 600 ampere-hours $(\mathrm{A \cdot h})$ of charge at 12.0 V. (a) Calculate the increase in rest mass of such a battery when it is taken from being fully depleted to being fully charged. (b) What percent increase is this, given the battery's mass is 20.0 kg?

Strategy

In part (a), we first must find the energy stored in the battery, which equals what the battery can supply in the form of electrical potential energy. Since $\mathrm{PE_{elec}} = qV$, we have to calculate the charge q in $600\ \mathrm{A \cdot h}$, which is the product of the current I and the time t. We then multiply the result by 12.0 V. We can then calculate the battery's increase in mass using $\Delta E = \mathrm{PE_{elec}} = (\Delta m)c^2$. Part (b) is a simple ratio converted to a percentage.

Solution for (a)

1. Identify the knowns. $I \cdot t = 600 \text{ A} \cdot \text{h}; V = 12.0 \text{ V}; c = 3.00 \times 10^8 \text{ m/s}$
2. Identify the unknown. Δm
3. Choose the appropriate equation. $\text{PE}_{\text{elec}} = (\Delta m)c^2$
4. Rearrange the equation to solve for the unknown. $\Delta m = \frac{\text{PE}_{\text{elec}}}{c^2}$
5. Plug the knowns into the equation.

$$\begin{aligned}\Delta m &= \frac{\text{PE}_{\text{elec}}}{c^2} \\ &= \frac{qV}{c^2} \\ &= \frac{(It)V}{c^2} \\ &= \frac{(600 \text{ A}\cdot\text{h})(12.0 \text{ V})}{(3.00\times10^8)^2}.\end{aligned} \tag{28.47}$$

Write amperes A as coulombs per second (C/s), and convert hours to seconds.

$$\begin{aligned}\Delta m &= \frac{(600 \text{ C/s}\cdot\text{h}\left(\frac{3600 \text{ s}}{1 \text{ h}}\right)(12.0 \text{ J/C})}{(3.00\times10^8 \text{ m/s})^2} \\ &= \frac{(2.16\times10^6 \text{ C})(12.0 \text{ J/C})}{(3.00\times10^8 \text{ m/s})^2}\end{aligned} \tag{28.48}$$

Using the conversion $1 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 1 \text{ J}$, we can write the mass as

$\Delta m = 2.88 \times 10^{-10} \text{ kg}.$

Solution for (b)

1. Identify the knowns. $\Delta m = 2.88 \times 10^{-10} \text{ kg}; m = 20.0 \text{ kg}$
2. Identify the unknown. % change
3. Choose the appropriate equation. $\% \text{ increase} = \frac{\Delta m}{m} \times 100\%$
4. Plug the knowns into the equation.

$$\begin{aligned}\% \text{ increase} &= \frac{\Delta m}{m} \times 100\% \\ &= \frac{2.88\times10^{-10} \text{ kg}}{20.0 \text{ kg}} \times 100\% \\ &= 1.44 \times 10^{-9}\%.\end{aligned} \tag{28.49}$$

Discussion

Both the actual increase in mass and the percent increase are very small, since energy is divided by c^2, a very large number. We would have to be able to measure the mass of the battery to a precision of a billionth of a percent, or 1 part in 10^{11}, to notice this increase. It is no wonder that the mass variation is not readily observed. In fact, this change in mass is so small that we may question how you could verify it is real. The answer is found in nuclear processes in which the percentage of mass destroyed is large enough to be measured. The mass of the fuel of a nuclear reactor, for example, is measurably smaller when its energy has been used. In that case, stored energy has been released (converted mostly to heat and electricity) and the rest mass has decreased. This is also the case when you use the energy stored in a battery, except that the stored energy is much greater in nuclear processes, making the change in mass measurable in practice as well as in theory.

Kinetic Energy and the Ultimate Speed Limit

Kinetic energy is energy of motion. Classically, kinetic energy has the familiar expression $\frac{1}{2}mv^2$. The relativistic expression for kinetic energy is obtained from the work-energy theorem. This theorem states that the net work on a system goes into kinetic energy. If our system starts from rest, then the work-energy theorem is

$$W_{\text{net}} = \text{KE}. \tag{28.50}$$

Relativistically, at rest we have rest energy $E_0 = mc^2$. The work increases this to the total energy $E = \gamma mc^2$. Thus,

$$W_{\text{net}} = E - E_0 = \gamma mc^2 - mc^2 = (\gamma - 1)mc^2. \tag{28.51}$$

Relativistically, we have $W_{\text{net}} = \text{KE}_{\text{rel}}$.

Relativistic Kinetic Energy

Relativistic kinetic energy is

$$\text{KE}_{\text{rel}} = (\gamma - 1)mc^2. \tag{28.52}$$

When motionless, we have $v = 0$ and

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = 1, \tag{28.53}$$

so that $\text{KE}_{\text{rel}} = 0$ at rest, as expected. But the expression for relativistic kinetic energy (such as total energy and rest energy) does not look much like the classical $\frac{1}{2}mv^2$. To show that the classical expression for kinetic energy is obtained at low velocities, we note that the binomial expansion for γ at low velocities gives

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2}. \tag{28.54}$$

A binomial expansion is a way of expressing an algebraic quantity as a sum of an infinite series of terms. In some cases, as in the limit of small velocity here, most terms are very small. Thus the expression derived for γ here is not exact, but it is a very accurate approximation. Thus, at low velocities,

$$\gamma - 1 = \frac{1}{2}\frac{v^2}{c^2}. \tag{28.55}$$

Entering this into the expression for relativistic kinetic energy gives

$$\text{KE}_{\text{rel}} = \left[\frac{1}{2}\frac{v^2}{c^2}\right]mc^2 = \frac{1}{2}mv^2 = \text{KE}_{\text{class}}. \tag{28.56}$$

So, in fact, relativistic kinetic energy does become the same as classical kinetic energy when $v << c$.

It is even more interesting to investigate what happens to kinetic energy when the velocity of an object approaches the speed of light. We know that γ becomes infinite as v approaches c, so that KE_{rel} also becomes infinite as the velocity approaches the speed of light. (See Figure 28.22.) An infinite amount of work (and, hence, an infinite amount of energy input) is required to accelerate a mass to the speed of light.

The Speed of Light

No object with mass can attain the speed of light.

So the speed of light is the ultimate speed limit for any particle having mass. All of this is consistent with the fact that velocities less than c always add to less than c. Both the relativistic form for kinetic energy and the ultimate speed limit being c have been confirmed in detail in numerous experiments. No matter how much energy is put into accelerating a mass, its velocity can only approach—not reach—the speed of light.

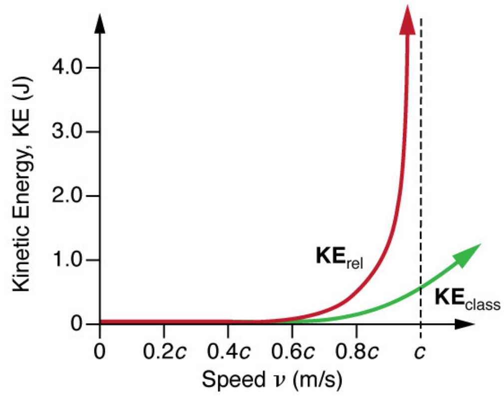

Figure 28.22 This graph of KE_{rel} versus velocity shows how kinetic energy approaches infinity as velocity approaches the speed of light. It is thus not possible for an object having mass to reach the speed of light. Also shown is KE_{class}, the classical kinetic energy, which is similar to relativistic kinetic energy at low velocities. Note that much more energy is required to reach high velocities than predicted classically.

EXAMPLE 28.8

Comparing Kinetic Energy: Relativistic Energy Versus Classical Kinetic Energy

An electron has a velocity $v = 0.990c$. (a) Calculate the kinetic energy in MeV of the electron. (b) Compare this with the classical value for kinetic energy at this velocity. (The mass of an electron is 9.11×10^{-31} kg.)

Strategy

The expression for relativistic kinetic energy is always correct, but for (a) it must be used since the velocity is highly relativistic (close to c). First, we will calculate the relativistic factor γ, and then use it to determine the relativistic kinetic energy. For (b), we will calculate the classical kinetic energy (which would be close to the relativistic value if v were less than a few percent of c) and see that it is not the same.

Solution for (a)

1. Identify the knowns. $v = 0.990c; m = 9.11 \times 10^{-31}$ kg
2. Identify the unknown. KE_{rel}
3. Choose the appropriate equation. $KE_{rel} = (\gamma - 1)mc^2$
4. Plug the knowns into the equation.
   First calculate γ. We will carry extra digits because this is an intermediate calculation.

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-\frac{(0.990c)^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-(0.990)^2}} \\ &= 7.0888\end{aligned} \tag{28.57}$$

   Next, we use this value to calculate the kinetic energy.

$$\begin{aligned}KE_{rel} &= (\gamma - 1)mc^2 \\ &= (7.0888 - 1)(9.11 \times 10^{-31}\text{ kg})(3.00 \times 10^8\text{ m/s})^2 \\ &= 4.99 \times 10^{-13}\text{ J}\end{aligned} \tag{28.58}$$

5. Convert units.

$$\begin{aligned}KE_{rel} &= (4.99 \times 10^{-13}\text{ J})\left(\frac{1\text{ MeV}}{1.60\times10^{-13}\text{ J}}\right) \\ &= 3.12\text{ MeV}\end{aligned} \tag{28.59}$$

Solution for (b)

1. List the knowns. $v = 0.990c; m = 9.11 \times 10^{-31}$ kg
2. List the unknown. $\mathrm{KE_{class}}$
3. Choose the appropriate equation. $\mathrm{KE_{class}} = \frac{1}{2}mv^2$
4. Plug the knowns into the equation.

$$\begin{aligned} \mathrm{KE_{class}} &= \tfrac{1}{2}mv^2 \\ &= \tfrac{1}{2}(9.00 \times 10^{-31}\ \mathrm{kg})(0.990)^2(3.00 \times 10^8\ \mathrm{m/s})^2 \\ &= 4.02 \times 10^{-14}\ \mathrm{J} \end{aligned} \quad (28.60)$$

5. Convert units.

$$\begin{aligned} \mathrm{KE_{class}} &= 4.02 \times 10^{-14}\ \mathrm{J}\left(\tfrac{1\ \mathrm{MeV}}{1.60 \times 10^{-13}\ \mathrm{J}}\right) \\ &= 0.251\ \mathrm{MeV} \end{aligned} \quad (28.61)$$

Discussion

As might be expected, since the velocity is 99.0% of the speed of light, the classical kinetic energy is significantly off from the correct relativistic value. Note also that the classical value is much smaller than the relativistic value. In fact, $\mathrm{KE_{rel}/KE_{class}} = 12.4$ here. This is some indication of how difficult it is to get a mass moving close to the speed of light. Much more energy is required than predicted classically. Some people interpret this extra energy as going into increasing the mass of the system, but, as discussed in Relativistic Momentum, this cannot be verified unambiguously. What is certain is that ever-increasing amounts of energy are needed to get the velocity of a mass a little closer to that of light. An energy of 3 MeV is a very small amount for an electron, and it can be achieved with present-day particle accelerators. SLAC, for example, can accelerate electrons to over $50 \times 10^9\ \mathrm{eV} = 50{,}000\ \mathrm{MeV}$.

Is there any point in getting v a little closer to c than 99.0% or 99.9%? The answer is yes. We learn a great deal by doing this. The energy that goes into a high-velocity mass can be converted to any other form, including into entirely new masses. (See Figure 28.23.) Most of what we know about the substructure of matter and the collection of exotic short-lived particles in nature has been learned this way. Particles are accelerated to extremely relativistic energies and made to collide with other particles, producing totally new species of particles. Patterns in the characteristics of these previously unknown particles hint at a basic substructure for all matter. These particles and some of their characteristics will be covered in Particle Physics.

Figure 28.23 The Fermi National Accelerator Laboratory, near Batavia, Illinois, was a subatomic particle collider that accelerated protons and antiprotons to attain energies up to 1 Tev (a trillion electronvolts). The circular ponds near the rings were built to dissipate waste heat. This accelerator was shut down in September 2011. (credit: Fermilab, Reidar Hahn)

Relativistic Energy and Momentum

We know classically that kinetic energy and momentum are related to each other, since

$$\mathrm{KE_{class}} = \frac{p^2}{2m} = \frac{(mv)^2}{2m} = \frac{1}{2}mv^2. \quad (28.62)$$

Relativistically, we can obtain a relationship between energy and momentum by algebraically manipulating their definitions. This produces

$$E^2 = (pc)^2 + (mc^2)^2, \tag{28.63}$$

where E is the relativistic total energy and p is the relativistic momentum. This relationship between relativistic energy and relativistic momentum is more complicated than the classical, but we can gain some interesting new insights by examining it. First, total energy is related to momentum and rest mass. At rest, momentum is zero, and the equation gives the total energy to be the rest energy mc^2 (so this equation is consistent with the discussion of rest energy above). However, as the mass is accelerated, its momentum p increases, thus increasing the total energy. At sufficiently high velocities, the rest energy term $(mc^2)^2$ becomes negligible compared with the momentum term $(pc)^2$; thus, $E = pc$ at extremely relativistic velocities.

If we consider momentum p to be distinct from mass, we can determine the implications of the equation $E^2 = (pc)^2 + (mc^2)^2$, for a particle that has no mass. If we take m to be zero in this equation, then $E = pc$, or $p = E/c$. Massless particles have this momentum. There are several massless particles found in nature, including photons (these are quanta of electromagnetic radiation). Another implication is that a massless particle must travel at speed c and only at speed c. While it is beyond the scope of this text to examine the relationship in the equation $E^2 = (pc)^2 + (mc^2)^2$, in detail, we can see that the relationship has important implications in special relativity.

Problem-Solving Strategies for Relativity

1. *Examine the situation to determine that it is necessary to use relativity.* Relativistic effects are related to $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, the quantitative relativistic factor. If γ is very close to 1, then relativistic effects are small and differ very little from the usually easier classical calculations.
2. *Identify exactly what needs to be determined in the problem (identify the unknowns).*
3. *Make a list of what is given or can be inferred from the problem as stated (identify the knowns).* Look in particular for information on relative velocity v.
4. *Make certain you understand the conceptual aspects of the problem before making any calculations.* Decide, for example, which observer sees time dilated or length contracted before plugging into equations. If you have thought about who sees what, who is moving with the event being observed, who sees proper time, and so on, you will find it much easier to determine if your calculation is reasonable.
5. *Determine the primary type of calculation to be done to find the unknowns identified above.* You will find the section summary helpful in determining whether a length contraction, relativistic kinetic energy, or some other concept is involved.
6. *Do not round off during the calculation.* As noted in the text, you must often perform your calculations to many digits to see the desired effect. You may round off at the very end of the problem, but do not use a rounded number in a subsequent calculation.
7. *Check the answer to see if it is reasonable: Does it make sense?* This may be more difficult for relativity, since we do not encounter it directly. But you can look for velocities greater than c or relativistic effects that are in the wrong direction (such as a time contraction where a dilation was expected).

CHECK YOUR UNDERSTANDING

A photon decays into an electron-positron pair. What is the kinetic energy of the electron if its speed is $0.992c$?

Solution

$$\begin{aligned} \mathrm{KE_{rel}} &= (\gamma - 1)\,mc^2 = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)mc^2 \\ &= \left(\frac{1}{\sqrt{1-\frac{(0.992c)^2}{c^2}}} - 1\right)(9.11\times10^{-31}\ \mathrm{kg})(3.00\times10^{8}\ \mathrm{m/s})^2 = 5.67\times10^{-13}\ \mathrm{J} \end{aligned} \tag{28.64}$$